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PERIODIC SOLUTIONS OF SYSTEMS WITH LAG CLOSELY RELATED TO LIAPUNOV SYSTEMS

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The familiar definition of Liapunov systems [1] is generalized for systems with lag. The present paper concerns a system closely related to that of Liapunov involving a small addition periodic in t. A theorem concerning the existence of a periodic solution is proved. An example is investigated.

1. Let us consider the system described by equations with lag of the form

$$\frac{dx}{dt} = \int_{-\tau}^{0} x(t+\vartheta)\, d\eta(\vartheta) + X(x(t+\vartheta)) + \mu F(t, x(t+\vartheta), \mu) \qquad (1.1)$$

where x is an n-dimensional vector and $\eta(\vartheta)$ is an $n \times n$ matrix of the functions $\eta_{ij}(\vartheta)$ with bounded variation defined on the segment $[-\tau, 0]$; the integral is to be interpreted in the Stieltjes sense; $X(x(\vartheta)) = \{X_i(x(\vartheta))\}$ is a nonlinear functional defined on the piecewise continuous functions $x(\vartheta)$, $-\tau \leqslant \vartheta \leqslant 0$ (with discontinuities of the first kind) bounded in norm, i.e. $\| x(\vartheta) \| < R$, where $R > 0$,

$$\| x(\vartheta) \| = \sup(| x_1(\vartheta)|, \ldots, | x_n(\vartheta)|), \quad -\tau \leqslant \vartheta \leqslant 0 \qquad (1.2)$$

Substituting any vector function $x(y, \vartheta)$ analytic in y and differentiable with respect to ϑ into the functional $X(x(\vartheta))$, we obtain the analytic function $X(x(y, \vartheta)) = X_1(y)$.

The functional $F(t, x(\vartheta), \mu) = \{F_i(t, x(\vartheta), \mu)\}$ (μ is a small parameter) is a continuous function of t, periodic with the period 2π, and is also a continuous function of μ; $|\mu| \leqslant \mu^*$, $\mu^* > 0$, $t \subset (-\infty, \infty)$.

Let us assume that the second Frechet derivative of the functional X and the first Frechet derivative of the functional F exist in some domain G of the space of piecewise-continuous functions $[-\tau, 0]$. This enables us to write

$$X(x(\vartheta) + z(\vartheta)) - X(x(\vartheta)) = X'(x(\vartheta))(z(\vartheta)) + \frac{1}{2}X''(x(\vartheta))(z(\vartheta), z(\vartheta)) + \omega_2(x(\vartheta), z(\vartheta)) \tag{1.3}$$

$$F(t, x(\vartheta) + z(\vartheta), \mu) - F(t, x(\vartheta), \mu) = F'(t, x(\vartheta), \mu)(z(\vartheta)) + \omega_1(t, x(\vartheta), z(\vartheta), \mu)$$

where X', F' are linear functionals of $z(\vartheta)$ and X'' is a quadratic functional of $z(\vartheta)$,

$$\lim_{\|z(\vartheta)\| \to 0} \frac{\|\omega_2(x(\vartheta), z(\vartheta))\|}{\|z(\vartheta)\|^2} = 0, \qquad \lim_{\|z(\vartheta)\| \to 0} \frac{\|\omega_1(t, x(\vartheta), z(\vartheta), \mu)\|}{\|z(\vartheta)\|} = 0$$

Further, let us assume that the derivatives X'', F' satisfy the Lipschitz conditions in $z(\vartheta)$ in the domain G.

Let us consider the "generating" system

$$\frac{dx}{dt} = \int_{-\tau}^{0} x(t+\vartheta)\, d\eta(\vartheta) + X(x(t+\vartheta)) \tag{1.4}$$

If we take the vector segment

$$x_t(\vartheta) = x(t+\vartheta) \qquad (-\tau \leqslant \vartheta \leqslant 0)$$

as an element of the solution, then system (1.4) is associated in the function space B of piecewise-continuous functions with norm (1.2) with the following system of "ordinary" differential equations with operator right side [2]:

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + R(x_t(\vartheta)) \tag{1.5}$$

$$Ax(\nu) = \left\{\frac{dx(\nu)}{d\nu} \quad \text{for} \quad -\tau \leqslant \nu < 0, \quad \int_{-\tau}^{0} x(\nu)\, d\eta(\nu) \quad \text{for} \quad \nu = 0\right\}$$

$$R(x(\nu)) = \{0 \quad \text{for} \quad -\tau \leqslant \nu < 0, \quad X(x(\nu)) \quad \text{for} \quad \nu = 0\}$$

Let the characteristic equation

$$\Delta(\lambda) \equiv \left| -E\lambda + \int_{-\tau}^{0} e^{\lambda\theta}\, d\eta(\vartheta) \right| = 0 \tag{1.6}$$

have the pair of purely imaginary roots $\lambda_{1,2} = \pm\, \omega i$.

Let us also assume that the remaining roots of characteristic equation (1.6) have negative real parts.

It is shown in [3] that in this case we can introduce the conjugate variables y, $\bar{y}$ and the vector function $z_{1t}(\vartheta)$ according to the formulas

$$y = f[x_t(\vartheta)], \quad \bar{y} = \bar{f}[x_t(\vartheta)], \quad z_{1t}(\vartheta) = x_t(\vartheta) - b(\vartheta)\, y - \bar{b}(\vartheta)\, \bar{y} \tag{1.7}$$

$$f[x_t(\vartheta)] = \sum_{j=1}^{n} \Delta_{jk_1}(\omega i) \left\{ -x_{jt}(0) + \sum_{l=1}^{n} \int_{-\tau}^{0} \left[\int_{0}^{\vartheta} x_{lt}(\xi) e^{\omega i(\vartheta-\xi)} d\xi \right] d\eta_{jl}(\vartheta) \right\} \quad (1.8)$$

$$b(\vartheta) = \{b_j(\vartheta)\} = \{\Delta_{l_1 j}(\omega i) e^{\omega i \vartheta}\} d^{-1}, \qquad d = \Delta_{l_1 k_1}(\omega i) \left[\frac{d\Delta(\lambda)}{d\lambda}\right]_{\lambda=\omega i} \quad (1.9)$$

Here $\Delta_{jl}(\omega i)$ is the algebraic complement of an element of the determinant $\Delta(\omega i)$ situated at the intersection of the jth row and the lth column $(\Delta_{l_1 k_1}(\omega i) \neq 0)$.

We can rewrite system (1.5) in the variables y, $\bar{y}$, $z_{1t}(\vartheta)$ to obtain (1.10)

$$dy / dt = \omega i y + Y_1(y, \bar{y}, z_{1t}(\vartheta)), \quad d\bar{y} / dt = -\omega i \bar{y} + \overline{Y}_1(\bar{y}, y, z_{1t}(\vartheta))$$

$$dz_{1t}(\vartheta) / dt = A z_{1t}(\vartheta) + Z_1(y, \bar{y}, z_{1t}(\vartheta)), \qquad z_{1t}(\vartheta) \subset L : f[z_{1t}(\vartheta)] = 0$$

Here Y_1, Z_1 satisfy all the requirements imposed on X; moreover, Z_1 assumes real values only. It is shown in [3] that the variables y, $\bar{y}$, $z_{1t}(\vartheta)$ have a one-to-one relationship with $x(\vartheta)$, so that systems (1.4) and (1.10) are equivalent.

We call system (1.10) a "Liapunov system with lag" if the following conditions are fulfilled:

1) characteristic equation (1.6) has no roots of the form $\pm N\omega i$ (where N is an integer, including zero) other than the pair of simple roots $\lambda_{1,2} = \pm \omega i$;

2) system (1.10) has a first integral of the form

$$y\bar{y} + S_1(y, \bar{y}, z_{1t}(\vartheta)) = \text{const} \quad (1.11)$$

where S_1 is an analytic function of y, $\bar{y}$ and a functional of $z_{1t}(\vartheta)$. The order of S_1 in all of its arguments in the neighborhood of the point $y = \bar{y} = z_{1t}(\vartheta) = 0$ is higher than two.

The existence of an integral of the form (1.11) implies that the simple case [3] of stability of system (1.10) applies.

Expressed in the variables y, $\bar{y}$, $z_{1t}(\vartheta)$ system (1.1) becomes

$$dy / dt = \omega i y + Y_1(y, \bar{y}, z_{1t}(\vartheta)) + \mu G_1(t, y, \bar{y}, z_{1t}(\vartheta), \mu) \quad (1.12)$$

$$d\bar{y} / dt = -\omega i \bar{y} + \overline{Y}_1(\bar{y}, y, z_{1t}(\vartheta)) + \mu \overline{G}_1(t, \bar{y}, y, z_{1t}(\vartheta), \mu)$$

$$dz_{1t}(\vartheta) / dt = A z_{1t}(\vartheta) + Z_1(y, \bar{y}, z_{1t}(\vartheta)) + \mu H_1(t, y, \bar{y}, z_{1t}(\vartheta), \mu)$$

where G_1, H_1 satisfy the same requirements as F and where H_1 assumes real values only.

We shall consider the existence and construction of periodic solutions for system (1.12) closely related to a Liapunov system with lag. The solutions to be considered become the periodic solution of generating system (1.10) for $\mu = 0$. A similar problem for a system containing terms with lag in small additions only was investigated in [4].

2. Generating system (1.10) has a family of periodic solutions [3] which depends on the arbitrary constants c and h,

$$y^\circ(t + h, c), \qquad \bar{y}^\circ(t + h, c), \qquad z_{1,t+h}^\circ(\vartheta, c) \quad (2.1)$$

with the initial conditions $y^\circ(0, c) = \bar{y}^\circ(0, c) = c$, where c is a constant of small absolute value. The period of solution (2.1) is given by the formula

$$T = \frac{2\pi}{\Omega(c)} = \frac{2\pi}{\omega}(1 + h_2 c^2 + h_4 c^4 + \cdots) \quad (2.2)$$

where series (2.2) contains terms with even powers of c only. We denote the first non-zero coefficient h_i by h_{2r}.

The relation

$$(1 + h_{2r}c^{2r} + \ldots) 2\pi / \omega = 2\pi / m \qquad (2.3)$$

where m is an arbitrary integer, enables us to determine those values of c for which solution (2.1) has the period 2π. When $h_{2r}(\omega - m) > 0$ Eq. (2.3) has only two real roots of which one is positive and the other negative [1]. Denoting one of these roots by c_m and substituting its value into (2.1), we obtain the solution

$$y^\circ(t + h, c_m), \quad \bar{y}^\circ(t + h, c_m), \quad z^\circ_{1,t+h}(\vartheta, c_m) \qquad (2.4)$$

with the period $2\pi / m$.

Let us return to system (1.10). We shall attempt to find a particular solution of system (1.1) in which $z_{1t}(\vartheta)$ can be expressed in series form,

$$z_{1t}^*(\vartheta, y, \bar{y}) = \sum_{k=1}^{\infty} z_{1t}^{(k)}(\vartheta, y, \bar{y}) \qquad (z_{1t}^{(k)}(\vartheta, y, \bar{y}) \subset L) \qquad (2.5)$$

Here $z_{1t}^{(k)}(\vartheta, y, \bar{y})$ is a kth order form in the variables $y, \bar{y}$.

It is easy to verify that in this case we have

$$\left(\frac{\partial z^{(k)}}{\partial y} y - \frac{\partial z^{(k)}}{\partial \bar{y}} \bar{y}\right) \omega i = Az^{(k)} + Q^{(k)}(\vartheta, y, \bar{y}) \quad (k \geqslant 1), \qquad Q^{(1)}(\vartheta, y, \bar{y}) \equiv 0 \qquad (2.6)$$

where $Q^{(k)}(\vartheta, y, \bar{y})$ is a known kth order form for a fixed k. Let

$$Q^{(k)}(\vartheta, y, \bar{y}) = \sum_{p+q=k} b_{pq}^{(k)}(\vartheta) y^p \bar{y}^q, \quad z^{(k)}(\vartheta, y, \bar{y}) = \sum_{p+q=k} a_{pq}^{(k)}(\vartheta) y^p \bar{y}^q \quad (2.7)$$

Let us substitute (2.7) into (2.6). Now, equating the coefficients of the product $y^p \bar{y}^q$, we obtain

$$[J(p - q)\omega i - A] \, a_{pq}^{(k)}(\vartheta) = b_{pq}^{(k)}(\vartheta) \qquad (J\varphi(\vartheta) \equiv \varphi(\vartheta)) \qquad (2.8)$$

Here J is an identity operator.

We note that the spectrum of the operator A does not contain any points of the form $\pm N\omega i$ (where N is an integer) in the subspace L. Hence, operator equation (2.8) is uniquely solvable for $a_{pq}^{(k)}(\vartheta)$, and

$$a_{pq}^{(k)}(\vartheta) \subset L, \quad a_{pq}^{(1)}(\vartheta) \equiv 0$$

In this way we obtain series (2.5) which formally satisfies system (1.10). In [3] it is shown that substitution of the functions $y^\circ, \bar{y}^\circ$ into series (2.5) yields a convergent series which is the periodic solution $z^\circ_{1,t+h}(\vartheta, c)$.

Let us set

$$z_{1t}(\vartheta) = z_t(\vartheta) + z_{1t}^*(\vartheta, y, \bar{y}) \qquad (2.9)$$

in system (1.10). This yields

$$dy / dt = \omega iy + Y(y, \bar{y}, z_t(\vartheta)), \qquad d\bar{y} / dt = -\omega i\bar{y} + \overline{Y}(\bar{y}, y, z_t(\vartheta)) \qquad (2.10)$$

$$dz_t(\vartheta) / dt = Az_t(\vartheta) + Z(y, \bar{y}, z_t(\vartheta), \vartheta) \qquad (Z(y, \bar{y}, 0, \vartheta) \equiv 0)$$

Periodic solution (2.4) of system (2.10) is

$$y^\circ(t + h, c_m), \ \bar{y}^\circ(t + h, c_m), \ z_t(\vartheta) \equiv 0 \qquad (2.11)$$

Let us construct the equations in variations of system (2.10) for solution (2.11),

$$\frac{d\xi}{dt} = \left[\omega i + \left(\frac{\partial Y}{\partial y}\right)_0\right] \xi + \left(\frac{\partial Y}{\partial \bar{y}}\right)_0 \bar{\xi} + Y'(y^\circ, \bar{y}^\circ, 0)(\eta_t(\vartheta)) \qquad (2.12)$$

$$\frac{d\bar{\xi}}{dt} = \left(\frac{\partial \bar{Y}}{\partial y}\right)_0 \xi + \left[-\omega i + \left(\frac{\partial \bar{Y}}{\partial \bar{y}}\right)_0\right]\bar{\xi} + \bar{Y}'(y^\circ, \bar{y}^\circ, 0)(\eta_t(\vartheta)) \qquad \text{(cont.)}$$

$$\frac{d\eta_t(\vartheta)}{dt} = A\eta_t(\vartheta) + Z'(y^\circ, \bar{y}^\circ, 0)(\eta_t(\vartheta))$$

Here and below the parentheses $(\ldots.)_0$ denote substitution of solution (2.11). It is easy to show that system (2.12) has a unique periodic solution of the form

$$\left\{\frac{dy^\circ}{dt} = \varphi,\ \frac{d\bar{y}^\circ}{dt} = \bar{\varphi},\ \eta_t(\vartheta) \equiv 0\right\} \qquad (2.13)$$

Let us construct the "conjugate" system

$$\frac{d\xi^*}{dt} = -\left[\omega i + \left(\frac{\partial Y}{\partial y}\right)_0\right]\xi^* - \left(\frac{\partial \bar{Y}}{\partial y}\right)_0 \bar{\xi}^*$$

$$\frac{d\bar{\xi}^*}{dt} = -\left(\frac{\partial Y}{\partial \bar{y}}\right)_0 \xi^* - \left[-\omega i + \left(\frac{\partial \bar{Y}}{\partial \bar{y}}\right)_0\right]\bar{\xi}^*$$

$$\frac{d\eta_t^*(\vartheta)}{dt} = -A^*\eta_t^*(\vartheta) - Z'^*(y^\circ, \bar{y}^\circ, 0)(\eta_t^*(\vartheta)) \qquad (2.14)$$

$$-A^*x(\vartheta) = \left\{\frac{dx_k(\vartheta)}{d\vartheta} \quad \text{for } 0 < \vartheta \leqslant \tau, \quad -\int_\tau^0 \sum_{j=1}^n x_j(\vartheta)\, d\eta_{\cdot k}(-\vartheta) \quad \text{for } \vartheta = 0\right\}$$

The operator Z'^* is defined in similar fashion. System (2.14) has a unique periodic solution

$$\{\psi, \bar{\psi}, \eta_t^*(\vartheta) \equiv 0\}$$

Let us determine ψ. Setting $\eta_t(\vartheta) = \eta_t^*(\vartheta) \equiv 0$ in (2.12) and (2.14), we obtain the following relation for the solutions:

$$\xi\xi^* + \bar{\xi}\bar{\xi}^* = \text{const} \qquad (2.15)$$

First integral (1.11) for system (2.10) can be written as

$$N = y\bar{y} + S(y, \bar{y}, z_t(\vartheta)) = \text{const} \qquad (2.16)$$

The order of the function S in all its arguments in the neighborhood of the point $y = \bar{y} = z_t(\vartheta) = 0$ is higher than two. It is easy to show that the expression

$$\left(\frac{\partial N}{\partial y}\right)_0 \xi + \left(\frac{\partial N}{\partial \bar{y}}\right)_0 \bar{\xi} + S'(y^\circ, \bar{y}^\circ, 0)(\eta_t(\vartheta)) = \text{const} \qquad (2.17)$$

is the first integral of system in variations (2.12).

Setting $\eta_t(\vartheta) \equiv 0$ in (2.17) and comparing with (2.15), we find that as our functions $\psi, \bar{\psi}$ we can take

$$\psi = \left(\frac{\partial N}{\partial y}\right)_0, \qquad \bar{\psi} = \left(\frac{\partial N}{\partial \bar{y}}\right)_0 \qquad (2.18)$$

Now let us consider system (1.12) after making substitution (2.9),

$$dy \,/\, dt = \omega i y + Y(y, \bar{y}, z_t(\vartheta)) + \mu G(t, y, \bar{y}, z_t(\vartheta), \mu)$$

$$d\bar{y} \,/\, dt = -\omega i \bar{y} + \bar{Y}(\bar{y}, y, z_t(\vartheta)) + \mu \bar{G}(t, \bar{y}, y, z_t(\vartheta), \mu)$$

$$dz_t(\vartheta) \,/\, dt = Az_t(\vartheta) + Z(y, \bar{y}, z_t(\vartheta), \vartheta) + \mu H(t, y, \bar{y}, z_t(\vartheta), \vartheta, \mu) \qquad (2.19)$$

Here the functional G and the operator H satisfy the same requirements as G_1, H_1. Let us introduce the notation

$$G_0 = G(t, y^\circ, \bar{y}^\circ, 0, 0), \qquad H_0 = H(t, y^\circ, \bar{y}^\circ, 0, 0)$$

$$Y_0'(x_t(\vartheta)) = Y'(y^\circ, \bar{y}^\circ, 0)(x_t(\vartheta)), \quad Z_0'(x_t(\vartheta)) = Z'(y^\circ, \bar{y}^\circ, 0)(x_t(\vartheta))$$

The function $w_t(\vartheta)$ is the unique periodic solution of the equation

$$\frac{dw_t(\vartheta)}{dt} = Aw_t(\vartheta) + Z_0'(w_t(\vartheta)) + H_0, \qquad w_t(\vartheta) \subset L$$

Theorem. System (2.19) has a periodic solution of period 2π which becomes generating solution (2.11) for $\mu = 0$ only if

$$P(h) \equiv \int_0^{2\pi} \left\{ [G_0 + Y_0'(w_t(\vartheta))] \left(\frac{\partial N}{\partial y}\right)_0 + [\bar{G}_0 + \bar{Y}_0'(w_t(\vartheta))] \left(\frac{\partial N}{\partial \bar{y}}\right)_0 \right\} dt = 0 \tag{2.20}$$

If, in addition,

$$\left(\frac{dP(h)}{dh}\right)_{h=h_1} \neq 0 \tag{2.21}$$

where $h = h_1$ is a root of Eq. (2.20), then the required periodic solution is unique. This solution is a continuous function of μ.

Proof. Making the substitution

$$y = y^\circ(t + h, \ c_m) + \mu u(t), \qquad z_t(\vartheta) = \mu v_t(\vartheta)$$

in system (2.19), we find that the latter becomes

$$\begin{aligned} \frac{du}{dt} &= \left[\omega i + \left(\frac{\partial Y}{\partial y}\right)_0\right] u + \left(\frac{\partial Y}{\partial \bar{y}}\right)_0 \bar{u} + Y_0'(v_t(\vartheta)) + G_0 + \mu\Phi \\ \frac{d\bar{u}}{dt} &= \left(\frac{\partial \bar{Y}}{\partial y}\right) u + \left[-\omega i + \left(\frac{\partial \bar{Y}}{\partial \bar{y}}\right)_0\right] \bar{u} + \bar{Y}_0'(v_t(\vartheta)) + \bar{G}_0 + \mu\bar{\Phi} \\ \frac{dv_t(\vartheta)}{dt} &= Av_t(\vartheta) + Z_0'(v_t(\vartheta)) + H_0 + \mu\chi \end{aligned} \tag{2.22}$$

Here

$$\begin{aligned} \Phi(t, u, \bar{u}, v_t(\vartheta), \mu) &= \frac{1}{2}\left(\frac{\partial^2 Y}{\partial y^2}\right)_0 u^2 + \left(\frac{\partial^2 Y}{\partial y \partial \bar{y}}\right)_0 u\bar{u} + \\ &+ \frac{1}{2}\left(\frac{\partial^2 Y}{\partial \bar{y}^2}\right)_0 \bar{u}^2 + \left(\frac{\partial G}{\partial y}\right)_0 u + \left(\frac{\partial G}{\partial \bar{y}}\right)_0 \bar{u} + \left(\frac{\partial G}{\partial \mu}\right)_0 + G_0'(v_t(\vartheta)) + \\ &+ \left(\frac{\partial Y'}{\partial y}\right)_0 (v_t(\vartheta)) u + \left(\frac{\partial \bar{Y}'}{\partial \bar{y}}\right)_0 (v_t(\vartheta)) \bar{u} + Y_0''(v_t(\vartheta), \ v_t(\vartheta)) + \mu(\ldots) \end{aligned} \tag{2.23}$$

and χ is defined in similar fashion.

Let us set $\mu = 0$ in system (2.22). The above equation now yields $v_t^{(0)}(\vartheta) = w_t(\vartheta)$, where $w_t(\vartheta) \subset L$. Let us substitute the vector function $v_t^{(0)}(\vartheta) = w_t(\vartheta)$ into the first two equations of system (2.22) for $\mu = 0$. The resulting system has a periodic solution u°, $\bar{u}^\circ$ only if (2.20). Let us assume that condition (2.20) has been satisfied by suitable choice of $h = h_1$. We can show that if, in addition, we have (2.21), then there exists a unique periodic solution of system (2.22). Along with system (2.22) we consider the ancillary system

$$\begin{aligned} \frac{du_1}{dt} &= \left[\omega i + \left(\frac{\partial Y}{\partial y}\right)_0\right] u_1 + \left(\frac{\partial Y}{\partial \bar{y}}\right)_0 \bar{u}_1 + Y_0'(v_{1t}(\vartheta)) + G_0 + \mu\Phi + W\varphi^* \\ \frac{d\bar{u}_1}{dt} &= \left(\frac{\partial \bar{Y}}{\partial y}\right)_0 u_1 + \left[-\omega i + \left(\frac{\partial \bar{Y}}{\partial \bar{y}}\right)_0\right] \bar{u}_1 + \bar{Y}_0'(v_{1t}(\vartheta)) + \bar{G}_0 + \mu\bar{\Phi} + W\bar{\varphi}^* \end{aligned} \tag{2.24}$$

$$\frac{dv_{1t}(\vartheta)}{dt} = Av_{1t}(\vartheta) + Z_0'(v_{1t}(\vartheta)) + H_0 + \mu\chi, \quad W = -\frac{\mu}{2\pi K}\int_0^{2\pi} [\Phi\psi + \bar{\Phi}\bar{\psi}) \, dt$$

$$\varphi^* = \left(\frac{\partial y^\circ}{\partial c} + \frac{t}{T}\frac{dT}{dc}\frac{\partial y^\circ}{\partial h}\right)_{c=c_m}, \qquad K = \int_0^{2\pi} [\varphi^*\psi + \bar{\varphi}^*\bar{\psi}] \, dt$$

The quantity T is defined by (2.2).

Ancillary system (2.24) is constructed in such a way that it always has a periodic solution. This solution can be obtained by the method of successive approximations. Let us take $W^{(0)} = 0$ and the functions u_1°, $\bar{u}_1^\circ$, $v_{1t}^{(0)}(\vartheta) = w_t(\vartheta)$ defined above as our first approximation. We can express the solution u_1° in the form

$$u_1^\circ = M\varphi + L_1(t,\ G_0 + Y_0'(w_t(\vartheta)),\ \bar{G}_0 + \bar{Y}_0'(w_t(\vartheta)))$$

Here L_1 is a completely defined bounded operator, M is an arbitrary constant, and the function φ is defined in (2.13).

The mth approximation can be determined from the system

$$\frac{du_1^{(m)}}{dt} = \left[\omega i + \left(\frac{\partial Y}{\partial y}\right)_0\right] u_1^{(m)} + \left(\frac{\partial Y}{\partial \bar{y}}\right)_0 \bar{u}_1^{(m)} + Y_0'(v_{1t}^{(m)}(\vartheta)) + G_0 + \mu\Phi^{(m-1)} + W^{(m)}\varphi^*$$

$$\frac{d\bar{u}_1^{(m)}}{dt} = \left(\frac{\partial \bar{Y}}{\partial y}\right)_0 u_1^{(m)} + \left[-\omega i + \left(\frac{\partial \bar{Y}}{\partial \bar{y}}\right)_0\right] \bar{u}_1^{(m)} + \bar{Y}_0'(v_{1t}^{(m)}(\vartheta) + \bar{G}_0 + \mu\bar{\Phi}^{(m-1)} + W^{(m)}\bar{\varphi}^* \qquad (2.25)$$

$$\frac{dv_{1t}^{(m)}(\vartheta)}{dt} = Av_{1t}^{(m)}(\vartheta) + Z_0'(v_{1t}^{(m)}(\vartheta)) + H_0 + \mu\chi^{(m-1)}$$

$$W^{(m)} = -\frac{\mu}{2\pi K}\int_0^{2\pi} [\Phi^{(m-1)}\psi + \bar{\Phi}^{(m-1)}\bar{\psi}]\, dt$$

Here the superscript $m-1$ accompanying the quantities Φ, χ denote the substitution of the $(m-1)$-th approximation. From system (2.25) we obtain

$$v_{1t}^{(m)}(\vartheta) = w_t(\vartheta) + L_2(t,\ \vartheta, \mu\ \chi^{(m-1)})$$

$$u_1^{(m)} = M\varphi + L_1(t,\ G_0 + Y_0'(v_{1t}^{(m)}(\vartheta)),\ \bar{G}_0 + \bar{Y}_0'(v_{1t}^{(m)}(\vartheta))) + L_1(t, \mu\Phi^{(m-1)} + W^{(m)}\varphi^*\ \mu\bar{\Phi}^{(m-1)} + W^{(m)}\bar{\varphi}^*)$$

where L_2 is a bounded operator.

Carrying out the appropriate estimates we can show that for a sufficiently small $|\mu|$ the sequences $\{u_1^{(m)}\}$, $\{W^{(m)}\}$, $\{v_{1t}^{(m)}(\vartheta)\}$ converge uniformly to certain functions $u_1^*(t, M, \mu)$, $W^*(M, \mu)$ and to the vector segment $v_{1t}^*(M, \vartheta, \mu)$.

System (2.22) has a periodic solution if and only if

$$W^*(M, \mu) = -\frac{\mu}{2\pi K}\int_0^{2\pi} [\Phi^*\psi + \bar{\Phi}^*\bar{\psi}]\, dt = 0$$

$$\Phi^* = \Phi(t,\ u_1^*,\ \bar{u}_1^*,\ u_{1t}^*(\vartheta),\ \mu) \qquad (2.26)$$

Recalling the form of the functional Φ (2.23) and carrying out some simple but cumbersome transformations, we can reduce Eq. (2.26) into

$$M\frac{dP(h)}{dh} + R + \mu(\ldots) = 0 \qquad (2.27)$$

where the function is defined in (2.20) and R is some completely defined constant. If condition (2.21) is fulfilled, then Eq. (2.27) is uniquely solvable for M for sufficiently small $|\mu|$ by virtue of the implicit function theorem; moreover, M is a continuous function of μ in the neighborhood of the point $\mu = 0$. This means that system (2.22), and therefore system (2.19), have a unique periodic solution.

Systems (2.19) and (1.1) are equivalent, which solves the problem of existence of a periodic solution of system (1.1).

The required periodic solution can be constructed by means of the same procedure as that used to prove the theorem. The difficulties having to do with the determination of the periodic solution of adjoint system (2.14) are avoided, since this solution is determined in the explicit form (2.18).

3. For example, let us consider the system described by the equation

$$\frac{dx(t)}{dt} = a_1 x(t) + a_2 x(t-\tau) + \gamma y_1^2 + \mu(a \sin mt + d \sin 2mt) \qquad (3.1)$$

where

$$a_1 = \omega \operatorname{ctg} \omega\tau, \quad a_2 = -\frac{\omega}{\sin \omega\tau}, \quad y_1 = \frac{\omega}{\sin \omega\tau} \int_{-\tau}^{0} x(t+\vartheta) \sin \omega(\tau+\vartheta)\, d\vartheta$$

It is easy to show that the generating system is (for $\mu = 0$) a Liapunov system with lag, since it has the first integral $\omega^2 y_1^2 + \dot{y}_1^2 - {}^2/_3 \gamma\omega y_1^3 = \text{const}$

and since the characteristic equation of the linear part of system (3.1) has a pair of purely imaginary roots $\lambda_{1,2} = \pm\omega i$, while the remaining roots have negative real parts.

The functional $f[x_t(\vartheta)]$ and the function $b(\vartheta)$ defined by formulas (1.8) and (1.9) in this case become

$$f[x(\vartheta)] \equiv x(0) + \frac{\omega}{\sin \omega\tau} e^{-i\omega\tau} \int_{0}^{-\tau} x(\xi) e^{-i\omega\xi}\, d\xi, \qquad b(\vartheta) = \frac{e^{i\omega\vartheta}}{1 - \omega\tau \csc \omega\tau e^{-i\omega\tau}}$$

Making substitutions (1.7), we can rewrite system (3.1) as

$$\frac{dy}{dt} = \omega i y + \left[\frac{\sqrt{\gamma}\omega}{\sin \omega\tau} \int_{-\tau}^{0} z_{1t}(\vartheta) \sin \omega(\tau+\vartheta)\, d\vartheta + \frac{\sqrt{\gamma}}{2} i(\bar{y} - y)\right]^2 + \mu(a \sin mt + d \sin 2mt)$$

$$\frac{d\bar{y}}{dt} = -\omega i \bar{y} + \left[\frac{\sqrt{\gamma}\omega}{\sin \omega\tau} \int_{-\tau}^{0} z_{1t}(\vartheta) \sin \omega(\tau+\vartheta)\, d\vartheta + \frac{\sqrt{\gamma}}{2} i(\bar{y} - y)\right]^2 + \mu(a \sin mt + d \sin 2mt)$$

$$\frac{dz_{1t}(\vartheta)}{dt} = Az_{1t}(\vartheta) + R[z_{1t}(\vartheta) + b(\vartheta) y + \bar{b}(\vartheta)\bar{y}] + \mu S(t, \vartheta) - \qquad (3.2)$$

$$- [b(\vartheta) + \bar{b}(\vartheta)] \left\{\left[\frac{\sqrt{\gamma}\omega}{\sin \omega\tau} \int_{-\tau}^{0} z_{1t}(\vartheta) \sin \omega(\tau+\vartheta)\, d\vartheta + \frac{\sqrt{\gamma}}{2} i(\bar{y} - y)\right]^2 + \mu(a \sin mt + d \sin 2mt)\right\}$$

where

$$Ax(\vartheta) = \left\{\frac{dx(\vartheta)}{d\vartheta} \quad \text{for} \quad -\tau \leqslant \vartheta < 0, \qquad a_1 x(0) + a_2 x(-\tau) \quad \text{for} \quad \vartheta = 0\right\}$$

$$R(x(\vartheta)) = \left\{0 \quad \text{for} \quad -\tau \leqslant \vartheta < 0, \ \frac{\gamma\omega^2}{\sin^2 \omega\tau}\left(\int_{-\tau}^{0} x(t+\vartheta) \sin \omega(\tau+\vartheta)\, d\vartheta\right)^2 \quad \text{for} \ \vartheta = 0\right\}$$

$$S(t, \vartheta) = \{0 \quad \text{for} \ -\tau \leqslant \vartheta < 0, \qquad \mu(a \sin mt + d \sin 2mt) \quad \text{for} \ \vartheta = 0\}$$

We shall attempt to find a particular solution $z_{1t}(\vartheta, y, \bar{y})$ of the form (2.5) for the generating system (system (3.2) for $\mu = 0$). Solving operator equations of the type (2.8), we obtain $z_{1t}^*(\vartheta, y, \bar{y}) = z_2(\vartheta, y, \bar{y}) + z_3(\vartheta, y, \bar{y}) + \ldots$

$$z_2(\vartheta, y, \bar{y}) = a_{20}(\vartheta) y^2 + a_{11}(\vartheta) y\bar{y} + a_{02}(\vartheta)\bar{y}^2 \qquad (3.3)$$

(cont.)

$$a_{20}(\vartheta) = -\frac{\gamma \sin \omega\tau e^{2i\omega\vartheta}}{4\omega\,[(\cos 2\omega\tau - \cos \omega\tau) + i\,(2\sin\omega\tau - \sin 2\omega\tau)} + \frac{\gamma}{4\omega i}\left(\frac{e^{i\omega\vartheta}}{K} + \frac{e^{-i\omega\vartheta}}{3\bar{K}}\right)$$

$$a_{11}(\vartheta) = \frac{\gamma \sin \omega\tau}{2\omega\,(1 - \cos\omega\tau)} + \frac{\gamma}{2\omega i}\left(\frac{e^{i\omega\vartheta}}{K} - \frac{e^{-i\omega\vartheta}}{\bar{K}}\right)$$

$$a_{02}(\vartheta) = \bar{a}_{20}(\vartheta), \qquad K = 1 - \frac{\omega\tau}{\sin\omega\tau}\, e^{-i\omega\tau}$$

Let us make the substitution

$$z_{1t}(\vartheta) = z_t(\vartheta) + z_{1t}^*(\vartheta, y, \bar{y}) \tag{3.4}$$

in system (3.2) and set $z_t(\vartheta) \equiv 0$ in the resulting generating system. This yields a system of ordinary second-order differential equations,

$$dy/dt = \omega i y + [{}^1/_2\sqrt{\gamma}\, i\,(\bar{y} - y) + \ldots]^2 \tag{3.5}$$

$$d\bar{y}/dt = -\omega i \bar{y} + [{}^1/_2\sqrt{\gamma}\, i\,(\bar{y} - y) + \ldots]^2$$

The terms not written out in the expressions in square brackets begin with a third-order form in y, $\bar{y}$, since simple computations show that

$$\int_{-\tau}^{0} a_{pq}(\vartheta) \sin\omega(\tau + \vartheta)\, d\vartheta = 0 \qquad (p + q = 2)$$

The substitution

$$y = u + y_2(u, \bar{u}) + y_3(u, \bar{u}) + \ldots$$

$$y_2(u, \bar{u}) = \frac{\gamma}{4\omega}\, i\,(u^2 + 2u\bar{u} - {}^1/_3\bar{u}^2)$$

$$y_3(u, \bar{u}) = -\frac{\gamma}{24\omega^2}(u^3 + 5u\bar{u}^2 + {}^1/_2\bar{u}^3) \tag{3.6}$$

transforms system (3.5) into

$$du/dt = \omega i u - {}^5/_{12}\gamma^2\omega^{-1}\, i u^2\bar{u} + \ldots \tag{3.7}$$

$$d\bar{u}/dt = -\omega i \bar{u} + {}^5/_{12}\gamma^2\omega^{-1}\, i u\bar{u}^2 - \ldots$$

System (3.7) has a family of periodic solutions

$$u = ce^{\varphi i}, \quad \varphi = \Omega(c)(t + h), \quad \Omega(c) = \omega - {}^5/_{12}\gamma^2\omega^{-1}c^2 + \ldots \tag{3.8}$$

which depends on the arbitrary constants c and h. The period of solution (3.8) is given by the formula

$$T(c) = \frac{2\pi}{\omega}\left(1 + \frac{5\gamma^2}{12\omega^2}c^2 + \ldots\right)$$

The equation $T(c) = 2\pi/m$ for $\omega > m$ yields the two real roots c_m. Generating solution (2.11) is of the form

$$y^\circ(t + h, c_m) = c_m e^{im(t+h)} + \frac{\gamma}{2\omega}\, i c_m^2\left(1 + \frac{1}{2}e^{2im(t+h)} - \frac{1}{6}e^{-2im(t+h)}\right) + c_m^3(\ldots) \tag{3.9}$$

The first integral of (2.16) for $z_t(\vartheta) \equiv 0$ is

$$H = y\bar{y} + \frac{\gamma i}{12\omega}(\bar{y} - y)^3 + \ldots = \text{const}$$

where the unwritten terms do not contain fourth-order terms in the variables y, $\bar{y}$.

Formula (2.18) yields $\psi = \bar{y}^0 - {}^1/_4\gamma i\omega^{-1}(\bar{y}^\circ - y^\circ)^2 + \ldots$

Equation (2.20) for determining h is

$$a \sin mh\left(1 - \frac{5\gamma^2}{24m^2}c^2 + \ldots\right) + \frac{d\gamma}{3m}c\cos 2mh + c^3(\ldots) = 0 \tag{3.10}$$

Solving Eq. (3.10) for h, we obtain

$$h = -\frac{d\gamma}{3am^2}c + c^3(\ldots)$$

Substituting the resulting value of h into formula (3.9) and the resulting expression for y° into formula (3.3), we obtain the first approximation of the required periodic solution of system (3.2). The formula

$$x_t^\circ(\vartheta) = z_{1t}^\circ(\vartheta) + b(\vartheta)y^\circ + \bar{b}(\vartheta)\bar{y}^\circ$$

yields the first approximation of the periodic solution of system (3.1). Computation of the subsequent approximation is not difficult. The expressions involved are extremely cumbersome, however.

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A STUDY OF NONLINEAR SYSTEMS OSCILLATIONS

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A procedure for investigating oscillations based on the small parameter method is described. The proposed procedure involves the use of nonlinear difference equations of special form. A mathematical justification of the procedure will be found in [1]. It consists essentially in the construction of an ancillary system of differential equations whose solution coincides at certain instants with the solution of the initial system. Applications considered include cases of resonance in quasilinear systems. A first-approximation integral stability criterion for periodic and almost-periodic solutions is derived.

1. The difference equations. Let us consider the following system of difference equations of order m:

$$X_{n+1} - X_n = \mu\Psi(X_n, n\mu, \mu) \qquad (n = 0, 1, 2, \ldots) \tag{1.1}$$

We assume that the right side is differentiable a sufficient number of times with respect to all of its arguments in some domain containing the solution X_n. We also assume that the parameter μ is small and that $\mu \geqslant 0$. Let us turn from (1.1) to a more general system of difference equations, introducing the ancillary vector function $Z(\tau, \mu)$ such that

$$Z(n\mu, \mu) = X_n \qquad (n = 0, 1, 2, \ldots) \tag{1.2}$$